ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 2, 1990

DIMENSION AND SUPERPOSITION OF CONTINUOUS FUNCTIONS

BY

MICHAEL LEVIN†

Department of Mathematics, University of Haifa, Haifa 31999, Israel

ABSTRACT

We give a relatively short proof of the following theorem of Sternfeld: Let X be a compact metric space with $\dim X \geqq 2$, and let $X \subset R^m$ be an embedding such that every $f \in C(X)$ can be represented as

$$f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} g_i(x_i), \qquad (x_1, x_2, \ldots, x_m) \in X, \quad g_i \in C(R).$$

Then $m \geqq 2 \dim X + 1$.

The theorem of Ostrand [3] (see also [4]), which generalizes the well known Kolmogorov's superposition theorem [2], says, in particular, that for every n-dimensional compact metric space X there exists an embedding $X \subset R^{2n+1}$ which satisfies the following: every $f \in C(X)$ can be represented as

$$f(x_1, x_2, \ldots, x_{2n+1}) = \sum_{i=1}^{2n+1} g_i(x_i), \qquad (x_1, x_2, \ldots, x_{2n+1}) \in X, \quad g_i \in C(R),$$

where $C(X)$ is the Banach space of real valued continuous functions on X.

It is clear that the number $2n+1$ is the best possible. (There are n-dimensional spaces which do not embed in R^{2n}.) Sternfeld [5] proved that it cannot be reduced for any n-dimensional compact metric space. In this paper we present a proof for this result of Sternfeld which is simpler than the original proof. The proof is based on ideas introduced in [5] (see [6] for a survey of related topics).

† This is a part of the author's Ph.D. thesis prepared at the University of Haifa under the supervision of Professor Y. Sternfeld.

Received November 10, 1989

Let X be a compact metric space. We identify the dual $C(X)^*$ of $C(X)$ with the space of real regular Borel measures on X with the total variation as norm.

Let $R_a = R$, $a \in N$ and let $A \subset N$ be a finite set. Denote $R_A = \Pi_{a \in A} R_a$ and let $R_\varnothing = 0 \in R$. Clearly R_A is homeomorphic to $R^{|A|}$, where $|A|$ is the cardinality of A. Let $P_B : R_A \to R_B$, $B \subset A$ denote the canonical projection. Let $X \subset R_A$, $\mu \in C(X)^*$ and $a \in A$. The measure μ_a in $C(R_a)^*$ is defined by $\mu_a(V) = \mu(P_a^{-1}(V) \cap X)$, $V \subset R_a$ a Borel set.

DEFINITION 1 [6]. Let X be a compact metric space. An embedding $X \subset R_A$ is said to be basic if every $f \in C(X)$ can be represented as $f(x) = \Sigma_{a \in A} g_a(x_a)$, $x = (x_a)_{a \in A} \in X$, $g_a \in C(R_a)$.

With this terminology the result which we intend to prove here is the following.

STERNFELD'S THEOREM [5]. *Let X be a compact metric space with* $\dim X > 2$, *and let $X \subset R_A$ be a basic embedding. Then* $|A| \geqq 2 \dim X + 1$.

We recall first the following result which gives a connection between basic embeddings and Borel measures.

THEOREM 1 [4]. *Let X be a compact metric space. $X \subset R_A$ is a basic embedding if and only if there exists a constant $\alpha > 0$, such that for each $\mu \in C(X)^*$*

$$\beta \geqq \alpha \| \mu \| , \tag{1}$$

where $\| \mu \|$ is the norm of μ and $\beta = \beta(\mu) =$ $\max\{ \| \mu_a \| : a \in A\}$.

Let $X \subset R_A$ and $K \in N$. Set

$$T_K = T_K(X \subset R_A) = \{B : B \subset A,\ |B| = K, \operatorname{int} P_B(V) \text{ is nonempty in } R_B \text{ for every nonempty open } V \text{ in } X\}$$

where $\operatorname{int} P_B(V)$ is the interior of $P_B(V)$.

Let $C^+(N)$ be the set of nonnegative real valued functions on N with a finite support and let 1_A denote the indicator function of A. Denote $I(f) = \Sigma_{i \in N} f(i)$, $f \in C^+(N)$. Obviously $I(cf) = cI(f)$, $I(f+g) = I(f) + I(g)$, $I(1_A) = |A|$, where $f, g \in C^+(N)$ and $c \in R$, $c \geqq 0$.

DEFINITION 2. Let $X \subset R_A$. $T_K = T_K(X \subset R_A)$ is said to be τ-full if 1_A can be represented as $1_A = f + \Sigma_{B \in T_K} \lambda_B 1_B$, where $\lambda_B \in R$, $\lambda_B \geqq 0$, $f \in C^+(N)$ and $I(f) = |A| - \tau$. If $\tau = |A|$ then T_K is said to be full.

It is clear that the fullness of T_K is equivalent to the existence of a representation of the form

$$1_A = \sum_{B \in T_K} \lambda_B 1_B, \qquad \lambda_B \geqq 0. \tag{2}$$

It is easy to check that the numbers λ_B can be chosen to be rationals. Hence (2) is equivalent to

$$n_A 1_A = \sum_{B \in T_K} n_B 1_B, \qquad n_A, n_B \in N, \quad n_A > 0. \tag{3}$$

We shall show in the second part of this paper that the assumption of existence of a basic embedding of a compact metric space $X \subset R_A$, with $|A| \leqq 2 \dim X$, implies the fullness of $T_{\dim X}(X \subset R_A)$. Then Sternfeld's theorem will follow from:

THEOREM 2. *Let X be a compact metric space, and let $X \subset R_A$ be a basic embedding. If $T_K = T_K(X \subset R_A)$ is full for some $K \geqq 2$, then $|A| \geqq 2K + 1$.*

PROOF. Assuming $|A| \leqq 2K$, we shall construct a set $\mathscr{E}$ of Borel measures on X with a finite support so that $\inf\{\beta(\mu)/\|\mu\| : \mu \in \mathscr{E}\} = 0$. This, by Theorem 1, contradicts the assumption that $X \subset R_A$ is a basic embedding and proves Theorem 2.

Let us first introduce some notation. Let Y, Z be finite subsets of X, so that $|Y| = |Z|$ and $Y \cap Z = \varnothing$. By (Y, Z) we denote the measure $\mu = \Sigma_{y \in Y} \delta_y - \Sigma_{z \in Z} \delta_z$, where δ_x is the Dirac measure with mass 1 at x. Clearly

$$\|(Y, Z)\| = |Y| + |Z| = 2|Y| = 2|Z|.$$

It is easy to verify that for each measure (Y, Z) on X and every $a \in A$ there exists a measure (Y^a, Z^a) on X such that

$$(Y, Z)_a = (Y^a, Z^a)_a, \qquad Y^a \subset Y, \quad Z^a \subset Z, \tag{4}$$

$$\|(Y^a, Z^a)\| = \beta((Y, Z)). \tag{5}$$

The fullness of T_K implies that T_K covers A. Applying this fact it is easy to check that for each nonempty open set V in X and every $l \in N$ there exist non-empty open sets $V_1, V_2, \ldots, V_l \in V$ so that

$$P_a(V_i) \cap P_a(V_j) = \varnothing, \qquad i \neq j, \quad a \in A, \tag{6}$$

and

(7) $$\prod_{a\in B} P_a\left(\bigcup_{i=1}^{l} V_i\right)\subset P_B(V), \qquad B\in T_K.$$

Assume now that a positive $c\in R$ and an $s\in N$ satisfy the following:

(8) For each nonempty open $U\subset X$ there exists a measure (Y^U, Z^U) on U so that

$$\beta((Y^U, Z^U))\leqq c\,\|(Y^U, Z^U)\|, \qquad |Y^U|=|Z^U|=s.$$

Let V be any empty open set in X, and let the nonempty open sets $V_1, V_2, \ldots, V_l\subset V$ satisfy (6) and (7), where $l\in N$ will be defined later on. Let (Y^{V_i}, Z^{V_i}), $i=1,2,\ldots,l$ be measures satisfying (8) for $U=V_i$. Our aim is to show that we can replace the pair c, s with a pair c', s' with c' smaller than c and thus by iteration we shall get the desired family $\mathscr{E}$. Since every open set contains two distinct points (by the fullness of T_k) it is evident that $c=s=1$ will satisfy (8). Set $Y=\bigcup_{i=1}^{l} Y^{V_i}$, $Z=\bigcup_{i=1}^{l} Z^{V_i}$. By (6) $V_i\cap V_j=\varnothing$ if $i\neq j$. Hence the measure (Y, Z) on V is well defined, and by (8) the following holds:

(9) $$\beta((Y,Z))\leqq c\,\|(Y,Z)\|, \qquad |Y|=|Z|=ls.$$

It is easy to check that from (6) it follows that

(10) $$|P_a(D)|\geqq\frac{|D\cap(Y\cup Z)|}{2s} \qquad \text{for every } D\subset X \text{ and each } a\in A.$$

Let (Y^a, Z^a), $a\in A$ be measures which satisfy (4) and (5). Denote $Y_0=Y$, $Z_0=Z$, $Y_0^a=Y^a$, $Z_0^a=Z^a$, $a\in A$. We shall construct, by induction on i, measures (Y_i, Z_i), (Y_i^a, Z_i^a), $a\in A$ on V. Assume that

$$(Y_{i-1}, Z_{i-1})_a=(Y_{i-1}^a, Z_{i-1}^a)_a, \qquad Y_{i-1}^a\subset Y_{i-1}, \quad Z_{i-1}^a\subset Z_{i-1}$$

(for $i=1$ this is satisfied), and let B_i be some element of T_K. Assume that

(11) $$\left|P_{B_i}(V)\cap\prod_{a\in B_i}P_a(Y_{i-1}^a)\right|>|P_{B_i}(Y_{i-1}\cup Z_{i-1})|.$$

Then, obviously, there is some $z\in V\backslash(Y_{i-1}\cup Z_{i-1})$ such that $P_a(z)\in P_a(Y_{i-1}^a)$ for each $a\in B_i$. It follows that $P_a(z)=P_a(y^a)$, $a\in B_i$ for some $y^a\in Y_{i-1}^a$. Similarly, if

(12) $$\left|P_{B_i}(V)\cap\prod_{a\in B_i}P_a(Z_{i-1}^a)\right|>|P_{B_i}(Y_{i-1}\cup Z_{i-1}\cup z)|,$$

then there there is $y \in V \setminus (Y_{i-1} \cup Z_{i-1} \cup z)$ and $z^a \in Z^a_{i-1}$, $a \in B_i$ such that $P_a(y) = P_a(z^a)$.

Define $Y_i = Y_{i-1} \cup y$, $Z_i = Z_{i-1} \cup z$ and for each $a \in A$

$$Y^a_i = \begin{cases} Y^a_{i-1} \setminus y_a, & a \in B_i, \\ Y^a_{i-1} \cup y, & a \notin B_i; \end{cases} \qquad Z^a_i = \begin{cases} Z^a_{i-1} \setminus z_a, & a \in B_i, \\ Z^a_{i-1} \cup z, & a \notin B_i, \end{cases}$$

It is easy to verify that the measures (Y_i, Z_i) and (Y^a_i, Z^a_i), $a \in A$ on V are well defined and that $(Y_i, Z_i)_a = (Y^a_i, Z^a_i)_a$, $Y^a_i \subset Y_i$, $Z^a_i \subset Z_i$,

$$|Y_i| = |Y_{i-1}| + 1, \quad |Z_i| = |Z_{i-1}| + 1,$$

$$|Y^a_i| = |Y^a_{i-1}| + (1_{A \setminus B_i} - 1_{B_i})(a) = |Y^a_{i-1}| + (1_A - 2 \cdot 1_{B_i})(a),$$

$$|Z^a_i| = |Z^a_{i-1}| + (1_A - 2 \cdot 1_{B_i})(a), \quad |Y^a_{i-1} \setminus Y^a_i| \leqq 1, \quad |Z^a_{i-1} \setminus Z^a_i| \leqq 1.$$

Assume now that this procedure can be carried out for $i = 1, 2, \ldots, t$ (i.e. we assume that (11) and (12) hold for $i = 1, 2, \ldots, t$). It follows from the above that

$$(Y_i, Z_i)_a = (Y^a_i, Z^a_i)_a, \tag{13}$$

$$|Y_i| = |Z_i| = |Y| + i = ls + i \qquad \text{(see (9))}, \tag{14}$$

$$\begin{aligned} |Y^a_i| = |Z^a_i| &= |Y^a| + (i \cdot 1_A - 2(1_{B_1} + 1_{B_2} + \cdots + 1_{B_i}))(a) \\ &= \tfrac{1}{2}\beta + f_i(a), \end{aligned} \tag{15}$$

where $f_i = i \cdot 1_A - 2(1_{B_1} + 1_{B_2} + \cdots + 1_{B_i})$ and

$$\beta = \beta((Y, Z)) = \| (Y^a, Z^a) \| = 2 |Y^a| \qquad \text{(see (5))},$$

$$|Y^a \setminus Y^a_i| \leqq i, \qquad |Z^a \setminus Z^a_i| \leqq i. \tag{16}$$

We return now to conditions (11) and (12). By (7) we have

$$P_{B_i}(V) \supset \prod_{a \in B_i} P_a\left(\bigcup_{j=1}^{l} V_j \right) \supset \prod_{a \in B_i} P_a(Y^a),$$

therefore

$$P_{B_i}(V) \cap \prod_{a \in B_i} P_a(Y^a_{i-1}) \supset \prod_{a \in B_i} P_a(Y^a \cap Y^a_{i-1}).$$

By (10) and (4)

$$|P_a(Y^a \cap Y^a_{i-1})| \geqq \frac{|(Y^a \cap Y^a_{i-1}) \cap (Y \cup Z)|}{2s} = \frac{|Y^a \cap Y^a_{i-1}|}{2s},$$

and by (16)

$$|Y^a \cap Y^a_{i-1}| \geqq |Y^a| - (i-1).$$

By (5) $|Y^a| = \frac{1}{2} \| (Y^a, Z^a) \| = \frac{1}{2}\beta((Y, Z)) = \frac{1}{2}\beta$. Since $X \subset R_A$ is a basic embedding $\beta \geqq \alpha \| (Y, Z) \|$ by Theorem 1. Hence, in view of (9),

$$|Y^a| = \tfrac{1}{2}\beta \geqq \tfrac{1}{2}\alpha \| (Y, Z) \| = \alpha l s.$$

Thus we have

$$\left| P_{B_i}(V) \cap \prod_{a \in B_i} P_a(Y^a_{i-1}) \right| \geqq \prod_{a \in B_i} \frac{|Y^a \cap Y^a_{i-1}|}{2s} \geqq \left(\frac{|Y^a| - (i-1)}{2s} \right)^{|B_i|}$$
$$\geqq \left(\frac{\alpha l s - i + 1}{2s} \right)^K$$

(since $B_i \in T_K$, so $|B_i| = K$).

Similarly, we obtain

$$\left| P_{B_i}(V) \cap \prod_{a \in B_i} P_a(Z^a_{i-1}) \right| \geqq \left(\frac{\alpha l s - i + 1}{2s} \right)^K.$$

On the other hand, by (14) we have

$$|P_{B_i}(Y_{i-1} \cup Z_{i-1})| \leqq |Y_{i-1}| + |Z_{i-1}| = 2ls + 2i - 2 < 2ls + 2i$$

and

$$|P_{B_i}(Y_{i-1} \cup Z_{i-1} \cup z)| \leqq |Y_{i-1}| + |Z_{i-1}| + 1 = 2ls + 2i - 1 < 2ls + 2i.$$

Hence from the condition

$$\left(\frac{\alpha l s - i + 1}{2s} \right)^K > 2ls + 2i$$

(11) and (12) follow. Since for any $i = 1, 2, \ldots, t$, $\alpha l s - i + 1 \geqq \alpha l s - t + 1$ and $2ls + 2t \geqq 2ls + 2i$, the condition

$$\text{(17)} \qquad \alpha l s - t \geqq 0, \qquad \left(\frac{\alpha l s - t + 1}{2s} \right)^K > 2ls + 2t$$

implies (11) and (12) for each $i = 1, 2, \ldots, t$. Hence if (17) holds, we can construct measures (Y_t, Z_t), (Y_t^a, Z_t^a), $a \in A$ which satisfy (13), (14), and (15).

Recall now that T_K is full, and therefore (3) holds. It follows that the sets B_i can be chosen in T_K so that $n_A 1_A = \Sigma_{i=1}^n 1_{B_i}$, where $n = \Sigma_{B \in T_K} n_B$ (see (3)). For $i > n$ define B_i by $B_i = B_{i \bmod n}$. It follows that for $t \equiv n \bmod n$,

$$f_t = t \cdot 1_A - 2(1_{B_1} + 1_{B_2} + \cdots + 1_{B_t}) = t \cdot 1_A - \frac{2n_A t}{n} 1_A = \left(t - \frac{2n_A t}{n}\right) 1_A.$$

Assume now that $|A| \leqq 2K$. Then by (3)

$$n_A |A| = n_A I(1_A) = I(n_A 1_A) = I\left(\sum_{B \in T_K} n_B 1_B\right) = \sum_{B \in T_K} n_B I(1_B) = K\left(\sum_{B \in T_K} n_B\right) = Kn,$$

and hence $n_A / n = K/|A| \geqq K/2K = \frac{1}{2}$. Therefore for $t \equiv n \bmod n$, $f_t = (t - 2n_A t/n) 1_A \leqq 0$, i.e. $f_t(a) \leqq 0$ for each $a \in A$.

Since $K \geqq 2$, it is easy to check that for a sufficiently large $l \in N$ we may choose t so that (17) holds and such that $t \geqq \frac{1}{2}\alpha ls$ and $t \equiv n \bmod n$. It follows that for this t the measures (Y_t, Z_t) and (Y_t^a, Z_t^a) can be constructed. Then by (15)

$$|Y_t^a| = |Z_t^a| = \tfrac{1}{2}\beta + f_t(a) \leqq \tfrac{1}{2}\beta$$

(since $f_t(a) \leqq 0$).

By (9) $\beta = \beta((Y, Z)) \leqq c \, \| (Y, Z) \| = 2cls$. Hence

$$\| (Y_t^a, Z_t^a) \| = |Y_t^a| + |Z_t^a| \leqq 2cls.$$

As $t \geqq \frac{1}{2}\alpha ls$ we get by (14)

$$\| (Y_t, Z_t) \| = |Y_t| + |Z_t| = 2(ls + t) \geqq 2(ls + \tfrac{1}{2}\alpha ls) = ls(2 + \alpha).$$

It follows that

$$\| (Y_t^a, Z_t^a) \| \leqq 2cls = c \frac{2}{2+\alpha} ls(2+\alpha) \leqq c \frac{2}{2+\alpha} \, \| (Y_t, Z_t) \| .$$

Obviously $\| \mu_\alpha \| \leqq \| \mu \|$ for any Borel measure μ. Therefore from (13) it follows that

$$\| (Y_t, Z_t)_a \| = \| (Y_t^a, Z_t^a)_a \| \leqq \| (Y_t^a, Z_t^a) \| .$$

Hence

$$\beta((Y_t, Z_t)) = \max\{ \| (Y_t, Z_t)_a \| : a \in A \} \leqq \max\{ \| Y_t^a, Z_t^a) \| : a \in A \}$$

$$\leqq c \frac{2}{2+\alpha} \| (Y_t, Z_t) \| .$$

We have shown that if (8) holds it does so also with $c' = c(2/2 + \alpha)$ and $s' = ls + t$. Therefore, by iteration there exists for each n a measure (Y_*, Z_*) such that $\beta((Y_*, Z_*)) \leqq (2/(2+\alpha))^n \| (Y_*, Z_*) \|$. As remarked in the beginning of the proof this verifies Theorem 2.

We turn now to the second part of this paper. Recall first the following facts from dimension theory (see [1]).

A finite-dimensional compact space X is called a Cantor manifold, if for all closed $F \subset X$ with $\dim F \leqq \dim X - 2$, $X \setminus F$ is connected. It is well known that each n-dimensional compact metric space contains some n-dimensional Cantor manifold. It is easy to show that for every closed subset F of a Cantor manifold X with $\operatorname{int} F \neq \varnothing$, $\dim F = \dim X$. A subset F of R^n is n-dimensional if and only if $\operatorname{int} F \neq \varnothing$.

The dimension $\dim f$ of a mapping $f: X \to Y$ is defined by $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$. The well known theorem of Hurewicz says, in particular, that for a continuous function $f: X \to Y$ of compact metric spaces $\dim X \leqq \dim Y + \dim f$. We begin with the following definition.

Definition 3. An embedding $X \subset R_A$ is said to be zero-dimensional if for each $B \subset A$, so that $|B| = |A| - 1$, $\dim(P_B |_X) = 0$, where $P_B |_X$ is the restriction of P_B to X.

Note that by $R_B \subset R_A$ we understand that $B \subset A$ and R_B is identified with $R_B^y = y \times R_B$ for some $y \in R_{A \setminus B}$. It is easy to prove

Proposition 1. *Let $X \subset R_A$ be a basic (zero-dimensional) embedding, and let $F \subset X \cap R_B \subset R_B \subset R_A$ be closed in X. Then the induced embedding $F \subset R_B$ is a basic (zero-dimensional) embedding.*

The following proposition shows that we may restrict ourselves to basic zero-dimensional embeddings.

Proposition 2 [5]. *Let $X \subset R_A$ be a basic embedding of an n-dimensional compact metric space X, where $n \geqq 2$. Then there exist some n-dimensional compact metric space F and some basic zero-dimensional embedding $F \subset R_B$ so that $|B| \leqq |A|$.*

PROOF. By Proposition 1, we may assume that X is a Cantor manifold. If $X \subset R_A$ is not zero-dimensional, then there exists some $b \in A$ so that $\dim P_B|_X > 0$, where $B = A \setminus b$. Hence there exists $R_b \subset R_A$ such that $\dim(R_b \cap X) \geqq 1$. It follows that $R_b \cap X$ contains a closed interval I. Let $F_1 = P_b^{-1}(I) \cap X$. Clearly $\operatorname{int} F_1 \neq \emptyset$, and since X is a Cantor manifold, $\dim F_1 = n \geqq 2$. It follows that $I \neq F_1$. Then there exists some closed $F_2 \subset F_1$ so that $F_2 \cap I = \emptyset$ and $\operatorname{int} F_2 \neq \emptyset$. Hence $\dim F_2 = n$, and we claim that $P_B|_{F_2}$ is an injection. Indeed, if not, then there exist $x_1, x_2 \in F_2$ so that $P_B(x_1) = P_B(x_2)$. Let $x_4 = P_b(x_1) \in I$ and $x_3 = P_b(x_2) \in I$. Set $\mu = \delta_{x_1} - \delta_{x_2} + \delta_{x_3} - \delta_{x_4}$. It is easy to check that $\beta(\mu) = 0$, and by Theorem 1 we obtain a contradiction.

Let $F = P_B(F_2)$. Then $\dim F = \dim F_2 = n$, and we shall show now that the embedding $F \subset R_B$ is basic. Given $\varphi \in C(F)$, define $f \in C(F_2 \cup I)$ as follows: $f(x) = 0$ if $x \in I$ and $f(x) = \varphi(P_B(x))$ if $x \in F_2$. Clearly $F_2 \cup I \subset R_A$ is basic. Hence there are $g_a \in C(R_a)$, $a \in A$, such that for all $x = (x_a)_{a \in A} \in F_2 \cup I$, $f(x) = \Sigma_{a \in A}\, g_a(x_a)$.

It is easy to check that $g_b|_I \equiv \text{const}$. Hence the functions g_a can be chosen such that $g_b \equiv 0$. Then $\varphi(x) = \Sigma_{a \in B}\, g_a(x_a)$ for $x = (x_a)_{a \in B} \in F$, i.e., $F \subset R_B$ is a basic embedding.

This procedure can be continued till we end up with a basic zero-dimensional embedding. This proves Proposition 2.

DEFINITION 4. An embedding $X \subset R_A$ is said to be reduced if the following hold:

(a) X is a Cantor manifold;
(b) $\operatorname{int} P_a(X) \neq \emptyset$ in R_a for every $a \in A$;
(c) if $\operatorname{int} P_B(X) \neq \emptyset$ in R_B, $B \subset A$, then $\operatorname{int} P_B(V) \neq \emptyset$ in R_B, for each nonempty open V in X.

The next proposition motivates the introduction of this definition.

PROPOSITION 3 [4]. *Let $X \subset R_A$ be a compact metric space. Then there exists $F \subset X \cap R_B \subset R_B \subset R_A$ so that $\dim F = \dim X$, and the embedding $F \subset R_B$ is reduced.*

PROOF. Let $n = \dim X$, and let $X_1 \subset X$ be an n-dimensional Cantor manifold. If $\operatorname{int} P_a(X) = \emptyset$ for some $a \in A$, then $X_1 \subset R_{A_1}$ for some $R_{A_1} \subset R_A$, $A_1 = A \setminus a$, and we reduce $X \subset R_A$ to the embedding $X_1 \subset R_{A_1}$.

On the other hand if $\operatorname{int} P_B(X_1) \neq \emptyset$, for $B \subset A$, and $\operatorname{int} P_B(V) = \emptyset$ for some open $V \subset X_1$, then for every n-dimensinal Cantor manifold $X_2 \subset V$, $\operatorname{int} P_B(X_2) = \emptyset$ and we reduce $X_1 \subset R_A$ to the embedding $X_2 \subset R_A$.

This procedure can be continued and, since it must end after finitely many steps, we shall end up with a reduced embedding. This proves Proposition 3.

Denote $\gamma_n = \min\{|A|$: there exists a reduced basic zero-dimensional embedding $X \subset R_A$ so that $\dim X = n\}$.

From Propositions 1, 2 and 3 it follows that we may restrict ourselves to reduced basic zero-dimensional embeddings (r.b.z. embeddings in short), i.e. Sternfeld's theorem is equivalent to $\gamma_n \geqq 2n+1$, $n \geqq 2$.

PROPOSITION 4 [5]. $\gamma_{n+1} > \gamma_n$.

PROOF. Let $\dim X = n+1$, and let $X \subset R_A$ be an r.b.z. embedding with $|A| = \gamma_{n+1}$. Since $\dim P_a(X) \leqq 1$ for every $a \in A$, it follows from Hurewicz's theorem that $\dim P_a |_X \geqq n$, and hence there exists some $R_B \subset R_A$, $B = A \setminus a$ so that $\dim(X \cap R_B) \geqq n$. By Propositions 1 and 3 there is some $F \subset X \cap R_{B_1} \subset R_{B_1} \subset R_B$ so that $\dim F = \dim(X \cap R_B) \geqq n$, and $F \subset R_{B_1}$ is an r.b.z. embedding. Since $|B_1| \leqq |B| = |A| - 1 < \gamma_{n+1}$, $\dim F \leqq n$. Hence $\dim F = n$, and $\gamma_n \leqq |B_1| < \gamma_{n+1}$. This proves Proposition 4.

PROPOSITION 5. $\gamma_1 \geqq 2$, $\gamma_2 \geqq 4$.

PROOF. If $\gamma_1 = 1$ then there is an r.b.z. embedding $X \subset R_a$, $a \in N$ with $\dim X = 1$. Hence $\dim P_\phi |_x = 1$, and therefore the embedding $X \subset R_a$ is not zero-dimensional. Thus $\gamma_1 \geqq 2$.

By Proposition 4, $\gamma_2 \geqq 3$. Assume that there exists some r.b.z. embedding $X \subset R_A$ so that $\dim X = 2$ and $A = \{a_1, a_2, a_3\}$. Then $\dim P_B |_X = 0$ for every $B \subset A$ with $|B| = 2$. From Hurewicz's theorem it follows that $\dim P_B(X) \geqq 2$, and hence $\operatorname{int} P_B(X) \neq \varnothing$ in R_B. Since $X \subset R_A$ is reduced, $B \in T_2 = T_2(X \in R_A)$. Therefore $T_2 = \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$, and we obtain

$$1_A = \tfrac{1}{2}\, 1_{\{a_1,a_2\}} + \tfrac{1}{2}\, 1_{\{a_1,a_3\}} + \tfrac{1}{2}\, 1_{\{a_2,a_3\}},$$

i.e. T_2 is full. By Theorem 2, this implies $|A| \geqq 4$ and contradicts the assumption that $|A| = 3$. Hence $\gamma_2 \geqq 4$, and Proposition 5 is proved.

PROPOSITION 6. *Let $X \subset R_A$ be an r.b.z. embedding so that* $\dim X = 1$. *Then $T_1(X \subset R_A)$ is 2-full.*

PROOF. As $\gamma_1 \geqq 2$, $|A| \geqq 2$. Let $A = \{a_1, a_2, \ldots\}$. The embedding $X \subset R_A$ is reduced. It follows that $\operatorname{int} P_a(X) \neq \varnothing$ for each $a \in A$.

Hence $T_1(X \subset R_A) = \{\{a_1\}, \{a_2\}, \ldots\}$. Let $1_A = f + 1_{a_1} + 1_{a_2}$. Obviously $f \in C^+(N)$ and $I(f) = |A| - 2$. This proves Proposition 6.

Consider the following statements.

(M_K): $\gamma_K \geqq 2K$.

(H_K): $T_K(X \subset R_A)$ is $2K$-full for each r.b.z. embedding $X \subset R_A$ with $\dim X = K$.

By Propositions 5 and 6, the statements M_1, M_2 and H_1 hold. The following proposition will be proved later.

Proposition 7. *The statements* M_K, H_K *and* M_{K+1} *imply* H_{K+1}.

We shall show now that the following theorem, which is equivalent to Sternfeld's theorem, follows from Proposition 7.

Theorem 3. $\gamma_n \geqq 2n+1$ *for* $n \geqq 2$.

Proof. By Proposition 7, M_1, H_1 and M_2 imply H_2. If $\gamma_2 = 4$, then there is an r.b.z. embedding $X \subset R_A$ so that $|A| = 4$. Since H_2 holds and $|A| = 4$, $T_2(X \subset R_A)$ is full and, by Theorem 2, $|A| > 4$. This contradiction proves that $\gamma_2 \geqq 5$.

By Proposition 4, $\gamma_3 \geqq 6$, i.e. M_3 holds. Similarly, since M_2, H_2 and M_3 hold, by Proposition 7 H_3 holds as well, and by Theorem 2, $\gamma_3 \geqq 7$. This procedure can be continued, and we obtain $\gamma_n \geqq 2n+1$ for each $n \geqq 2$, and Theorem 3 is proved.

For the proof of Proposition 7 we need the following.

Lemma 1 [5]. *Let* $X \subset R_A$ *be a reduced embedding, and let* $K = \dim X - 1 \geqq 0$. *Then for each* $a \in A$ *there exist* $B_a \subset A \setminus a$ *and* $F_a \subset X \cap R_{B_a} \subset R_{B_a} \subset R_A$, *with* $\dim F_a = K$, *so that the embedding* $F_a \subset R_{B_a}$ *is reduced, and* $a \cup B \in T_{K+1}(X \subset R_A)$ *for every* $B \in T_K(F_a \subset R_{B_a})$.

Proof. Let $y \in \operatorname{int} P_a(X)$, $a \in A$, and set $B = A \setminus a$. Since X is a Cantor manifold, $\dim(X \cap R_B^y) \geqq K$ (recall that $R_B^y = y \times R_B \subset R_A$). Pick some closed K-dimensional $X_y \subset X \cap R_B^y$. By Proposition 3 there exists some $F_y \subset X_y \cap R_{B_y} \subset R_{B_y} \subset R_B^y$ so that $F_y \subset R_{B_y}$ is reduced and $\dim F_y = K$. Set $T = \{B : |B| = K, B \subset A\}$. As the set A is finite, T is finite too. Let $T = \{B_1, B_2, \ldots, B_n\}$, and let $\{V_j^i\}_{j=1}^{\infty}$ be a basis for the topology of R_{B_i}, $i = 1, 2, \ldots, n$. Clearly $T_K = T_K(F_y \subset R_{B_y}) \subset T$, and therefore $T_K = \{B_{i_1}, B_{i_2}, \ldots, B_{i_t}\}$, where $t = t(y) = |T_K|$. It follows that $\operatorname{int} P_{B_{i_l}}(F_y) \neq \varnothing$, $l = 1, 2, \ldots, t$. Hence, there exists some j_l such that $V_{j_l}^{i_l} \subset \operatorname{int} P_{B_{i_l}}(F_y)$.

Let $S_y = \{i_1, j_1, i_2, j_2, \ldots, i_t, j_t\}$, $t = t(y)$. Obviously, the set of all disjoint sets S_y is countable. Hence there exists some $y \in \operatorname{int} P_a(V)$ so that

$$W_y = \{y' : y' \in \operatorname{int} P_a(X) \text{ and } S_{y'} = S_y\}$$

is of second category in $\operatorname{int} P_a(X)$, i.e., $V = \operatorname{int} \bar{W}_y \neq \emptyset$. The reader may easily verify that $\operatorname{int} P_{a \cup B_{i_l}}(X) \supset V \times V_{j_l}^{i_l}$, and since the embedding $X \subset R_A$ is reduced, $a \cup B_{i_l} \in T_{K+1}(X \subset R_A)$.

Set $F_a = F_y$, $B_a = B_y$, and this completes the proof of Lemma 1.

Let $A \subset N$ be a finite set, let $a \in A$, and let $f \in C^+(N)$. The pair (a, A) is said to be singular for f if $f(a) > 0$, and for each $B \subset N$ so that $a \in B \subset A$, $\Sigma_{b \in B} f(b) \geqq |B| f(d)$ for all $d \in A \setminus B$. Clearly if (a, A) is a singular pair for $f \in C^+(N)$, then for every B so that $a \in B \subset A$, the pair (a, B) is singular for f.

LEMMA 2. *Let $A \subset N$ be a finite set and let $f_a \in C^+(N)$, $a \in A$. If for each $a \in A$ the pair (a, A) is singular for f_a, then there exist reals $\lambda_a \geqq 0$ such that $1_A = \Sigma_{a \in B} \lambda_a (1_A \cdot f_a)$.*

PROOF. For $f \in C^+(N)$ set $\min f = \min\{f(a) : a \in A\}$, $M(f) = \{a : f(a) = \min f, a \in A\}$. Denote $G = \{\Sigma_{a \in A} \lambda_a (1_A \cdot f_a) : \lambda_a \geqq 0\} \subset C^+(N)$, $m(G) = \min\{|M(f)| : f \in G \text{ and } \min f \neq 0\}$.

Since $g_a(a) > 0$, $a \in A$, $\min(\Sigma_{a \in A} 1_A \cdot f_A\} \neq 0$. Therefore $m(G) \geqq 1$. Obviously, the lemma is satisfied for $|A| = 1$. Assume now that the lemma holds for $|A| = 1, 2, \ldots, n-1$, and we shall prove it for $|A| = n$.

Clearly if $m(G) = n$, then the lemma holds for $|A| = n$. Let $m(G) < n$. Then there exists some $f \in G$ so that $|M(f)| = m(G)$. Set $B = M(f)$. For every $a \in B$ the pair (a, B) is singular for f_a and by the assumption there exists reals $\lambda'_a \geqq 0$ so that $1_B = \Sigma_{a \in B} \lambda'_a (1_B \cdot f_a)$.

Denote $S = \Sigma_{a \in B} \lambda'_a (1_A \cdot f_a)$ and let $S_x = f + xS$ for $x \in R$, $x \geqq 0$. Obviously $S \in G$ and $S_x \in G$. Let $d \in A \setminus B$ and $a \in B$. Since (a, A) is singular for f_a, $\Sigma_{b \in B} f_a(b) \geqq |B| f_a(d)$.

From the above it follows that

$$|B| S(d) = |B| \left(\sum_{a \in B} \lambda'_a (1_A \cdot f_a) \right)(d) = \sum_{a \in B} \lambda'_a (|B| \cdot f_a(d)) \leqq \sum_{a \in B} \lambda'_a \left(\sum_{b \in B} f_a(b) \right)$$

$$= \sum_{b \in B} \left(\sum_{a \in B} \lambda'_a f_a(b) \right) = \sum_{b \in B} \left(\sum_{a \in B} \lambda'_a (1_B \cdot f_a) \right)(b) = \sum_{b \in B} 1_B(b) = |B|.$$

Hence $S(d) \leqq 1$ for $d \in A \setminus B$. If $S(d) = 1$ for all $d \in A \setminus B$, then $S = 1_A$ and $m(G) = |A| = n$. Hence there is some $d \in A \setminus B$ so that $S(d) < 1$. Let $b \in B$. Then $f(d) > f(b)$. Since $S_x(d) = f(d) + xS(d)$ and $S_x(b) = f(b) + xS(b) = f(b) + x$, there is $x \geqq 0$ so that $S_x(d) = S_x(b)$. It follows that for some $0 \leqq y \leqq x$, $|M(S_y)| \geqq |B| + 1$. This contradiction proves Lemma 2.

PROOF OF PROPOSITION 7. *Let $X \subset R_A$ be an r.b.z. embedding with $\dim X = K+1$. By Lemma 1, for each $a \in A$ there exists an r.b.z. embedding $F_a \subset R_{B_a}$ so that $B_a \subset A \setminus a$, $\dim F_a = K$ and $a \cup B \in T = T_{K+1}(X \subset R_A)$ for every $B \in T_a = T_K(F_a \subset R_{B_a})$.*

Recall that we assume M_K, H_K and M_{K+1}. Therefore

$$1_{B_a} = f_a + \varphi_a, \qquad I(f_a) = |B_a| - 2K, \tag{18}$$

where $f_a \in C^+(N)$ and $\varphi_a = \Sigma_{B \in T_a} \lambda_B^a 1_B$, $\lambda_B^a \geqq 0$. It follows that

$$I(\varphi_a) = I(1_{B_a}) - I(f_a) = |B_a| - (|B_a| - 2K) = 2K. \tag{19}$$

Set $G = \{\Sigma_{B \in T} \lambda_B 1_B : \lambda_B \geqq 0\} \subset C^+(N)$.

Denote $\tilde{\varphi}_a = \Sigma_{B \in T_a} \lambda_B^a 1_{a \cup B}$. Clearly $\tilde{\varphi}_a \in G$ and $\tilde{\varphi}_a(a) = \Sigma_{B \in T_a} \lambda_B^a$ and since

$$I(\varphi_a) = I\Big(\sum_{B \in T_a} \lambda_B^a 1_B\Big) = \sum_{B \in T_a} \lambda_B^a I(1_B) = K \sum_{B \in T_a} \lambda_B^a,$$

by (19) $\Sigma_{B \in T_a} \lambda_B^a = 2$, and we obtain $\tilde{\varphi}_a(a) = 2$. Now it is clear that

$$\tilde{\varphi}_a = \varphi_a + 2 \cdot 1_a. \tag{20}$$

Let $\psi_a = (|A| - 2K - 2)1_a$. Then, by M_{K+1}, $\psi_a \in C^+(N)$.

Set $\Phi_a = \tilde{\varphi}_a + \psi_a$. We shall show that the pair (a, A) is singular for Φ_a. Indeed, let $a \in Z \subset A$ and $b \in A \setminus Z$. Then by (18) and (20)

$$\begin{aligned}
&\sum_{z \in Z} \Phi_a(z) \\
&\quad = \sum_{z \in Z} \tilde{\varphi}_a(z) + \sum_{z \in Z} \psi_a(z) = \tilde{\varphi}_a(a) + \psi_a(a) + \sum_{z \in Z \cap B_a} \varphi_a(z) \\
&\quad = 2 + |A| - 2K - 2 + \sum_{z \in Z \cap B_a} \varphi_a(z) = |A| - |B_a| + I(f_a) + \sum_{z \in Z \cap B_a} \varphi_a(z) \\
&\quad \geqq |A| - |B_a| + \sum_{z \in Z \cap B_a} f_a(z) + \varphi_a(z) = |A| - |B_a| + |Z \cap B_a| \geqq |Z| \\
&\quad \geqq |Z|(f_a(b) + \varphi_a(b)) \geqq |Z|\varphi_a(b) = |Z|\Phi_a(b).
\end{aligned}$$

Hence the pair (a, A) is singular for Φ_a. Since $1_A \cdot \Phi_a = \Phi_a$, it follows from Lemma 2 that there are reals $\lambda_a \geqq 0$ so that $1_A = \Sigma_{a \in A} \lambda_a \Phi_a$. Let $\varphi = \Sigma_{a \in A} \lambda_a \tilde{\varphi}_a$ and $\psi = \Sigma_{a \in A} \lambda_a \psi_a$.

Obviously $\psi \in C^+(N)$, $\varphi \in G$ and $1_A = \psi + \varphi$. By (18), (19), and (20) we obtain

$$|A| = I(1_A) = I(\psi + \varphi) = I(\psi) + I(\varphi) = \sum_{a \in A} \lambda_a I(\psi_a) + \sum_{a \in A} \lambda_\alpha I(\tilde{\varphi}_a)$$

$$= \sum_{a \in A} \lambda_a(|A| - 2K - 2) + \sum_{a \in A} \lambda_a(2 + I(\varphi_a))$$

$$= \sum_{a \in A} \lambda_a(|A| - 2K - 2) + \sum_{a \in A} \lambda_a(2 + 2K) = |A| \sum_{a \in A} \lambda_a.$$

Hence

$$\sum_{a \in A} \lambda_a = 1$$

and

$$I(\psi) = \sum_{a \in A} \lambda_a I(\psi_a) = \sum_{a \in A} \lambda_a(|A| - 2K - 2) = |A| - 2K - 2.$$

Thus T is $2(K+1)$-full, and the proposition follows.

References

1. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1974.

2. A. N. Kolmogorov, *On the representation of continuous functions of many variables by superpositions of continuous functions of one variable and addition*, Dokl. Akad. Nauk SSSR **114** (1957), 953–956; Am. Math. Soc. Transl. (2) **28** (1963), 55–61.

3. P. A. Ostrand, *Dimension of metric spaces and Hilbert's problem 13*, Bull. Am. Math. Soc. **71** (1965), 619–622.

4. Y. Sternfeld, *Superpositions of continuous functions*, J. Approx. Theory **22** (1979), 360–368.

5. Y. Sternfeld, *Dimension, superposition of functions and separation of points, in compact metric spaces*, Isr. J. Math. **50** (1985), 13–53.

6. Y. Sternfeld, *Hilbert's 13th problem and dimension*, in *Geometric Aspects of Functional Analysis* (J. Lindenstrauss and V. Milman, eds.), Lecture Notes in Math. **1376**, Springer-Verlag, Berlin, 1987–8, pp. 1–49.